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USE OF RECURSION RELATIONS TO COMPUTE ONE-LOOP HELICITY AMPLITUDES*

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ABSTRACT

We illustrate the use of recursion relations in the computation of certain one-loop helicity amplitudes containing an arbitrary number of gauge bosons. After a brief review of the recursion relations themselves, we discuss the resolution of the apparent conflict between the spinor helicity method used to solve the recursion relations and the dimensional regulator used in the loop integrals. We then outline the procedure for constructing loop amplitudes, and present two examples of results obtained in this manner.

This talk will describe the use of recursion relations to compute one-loop corrections to multiple gauge boson scattering. A different approach to the same problem has been discussed by Dixon at this conference.[1]

We will begin with a brief review of the recursion relation for the double-off-shell quark current.[2,3] We define the double-off-shell quark current $\widehat{\Psi}_{ji}(\mathcal{Q};1,\ldots,n)$ to consist of the sum of all tree diagrams containing exactly one (massless) quark line with n on-shell gluons attached in all possible ways. The quark has momentum $\mathcal{P}$ and color index i, the antiquark momentum $\mathcal{Q}$ and color index j, and the ℓth gluon has momentum k_ℓ and color index a_ℓ. We take all of the momenta to flow into the diagram: $\mathcal{P}+\mathcal{Q}+k_1+\cdots+k_n=0$. Berends and Giele have shown that[2]

$$\hat{\Psi}_{ji}(\mathcal{Q};1,\ldots,n) = g^n \sum_{\mathcal{P}(1\ldots n)} [T^{a_1}\cdots T^{a_n}]_{ji}\Psi(\mathcal{Q};1,\ldots,n) \tag{1}$$

where g is the gauge coupling, T^a is a representation matrix for $SU(N)$, $\Psi(\mathcal{Q};1,\ldots,n)$ is the color-ordered current, and the notation $\mathcal{P}(1\ldots n)$ tells us that the sum runs over all permutations of the gluon labels $\{1,\ldots,n\}$. The color-ordered current contains only kinematic factors, and satisfies the recursion relation

$$\Psi(\mathcal{Q};1,\ldots,n) = -\sum_{j=0}^{n-1}\Psi(\mathcal{Q};1,\ldots,j)\ \not{J}(j{+}1,\ldots,n)\frac{\not{\mathcal{Q}}+\not{k}_1+\cdots+\not{k}_n}{[\mathcal{Q}+k_1+\cdots+k_n]^2}, \tag{2}$$

where $J(j{+}1,\ldots,n)$ is the color-ordered gluon current, which is derived from the sum of all tree graphs containing exactly $n{-}j$ on-shell gluons plus one off-shell gluon.[2]

The recursion relation (2) is easily solved for the case of n like-helicity gluons using an appropriate spinor-helicity basis for the gluon polarization vectors.[2] Rather than go into the details of the solution, let us note two of its features. First, it consists of a sum of terms containing only two propagator factors each, instead of the maximum of n that might be expected. Second, for large $\mathcal{Q}$, it falls off as $1/\mathcal{Q}^3$. Hence, any integrals over $\mathcal{Q}$ involving just the current and no additional inverse powers of $\mathcal{Q}$ are ultraviolet divergent, bringing up the question of an appropriate regulator.

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The (apparent) difficulty may be summarized as follows: the solution of the recursion relation relies heavily upon the use of a spinor helicity basis for the gluon polarization vectors. Thus, the chiral projectors $\frac{1}{2}(1 \pm \gamma_5)$ play an important role. Unfortunately, the only viable regulator for QCD is dimensional regularization, which does not allow for a consistent definition of γ_5. Fortunately, it is possible to separate the problem into two pieces in such a way that a d-dimensional definition of γ_5 is not required.

The variant of dimensional regularization which we employ is the original implementation by 't Hooft and Veltman[4] in which *only* the loop momentum is continued to d dimensions; all external quantities remain in four dimensions. Denoting the d-dimensional loop momentum by $\mathcal{L}$, we may write $\mathcal{L} \equiv L + m$, where L contains only the usual four space-time components of the momentum, and m contains only the "extra" components generated by the continuation to d dimensions. Setting $m^2 \equiv -\mu^2$, this decomposition implies that $L \cdot m = 0$, $\mathcal{L}^2 = L^2 - \mu^2$, and $\mathcal{L} \cdot q = L \cdot q$, where q is any external vector. Furthermore, $\not{m}$ anticommutes with $\not{q}$ and $\not{L}$ (but not $\not{\mathcal{L}}$). At one loop, it is clear that the momentum shifts required to do the integration involve only the first four components of $\mathcal{L}$. Hence, any term containing an odd number of factors of m integrates to zero.

Based on these properties, we can take any expression we would have written for a particular amplitude using the traditional Feynman rules and rewrite it as a sum of terms whose numerator dependence on the loop momentum has the form $L^{\nu_1} \cdots L^{\nu_i} \mu^{2j}$. Since only four-dimensional vectors are contracted into the Dirac matrices in this form, we may translate these expressions into spinor helicity notation and solve the recursion relation as usual. The only difference is that we must keep track of any additional μ^2-containing terms which may be generated along the way.[a] This procedure is only practical if many of these "extra" terms vanish, and there is a simple prescription for identifying those terms which do not vanish.

Consider the loop integral which contains m powers of μ^2 and n denominators:

$$\mathcal{J} \equiv \int \frac{d^d \mathcal{L}}{(2\pi)^d} \frac{\mu^{2m}}{\mathcal{L}^2 [\mathcal{L}+q_1]^2 [\mathcal{L}+q_2]^2 \cdots [\mathcal{L}+q_{n-1}]^2}. \tag{3}$$

From the discussion below, it will be obvious how to handle the case including powers of L. We may introduce Feynman parameters and carry out the momentum integration by writing $d^d\mathcal{L} = d^4 L d^{d-4} m$, with the result

$$\mathcal{J} = -\frac{i(-1)^n}{16\pi^2} \frac{\epsilon (4\pi)^\epsilon}{\Gamma(1-\epsilon)} \Gamma(m-\epsilon)\Gamma(n-m-2+\epsilon) \int_0^\infty d^n z\, \delta(1-\Sigma z) [-f(q_i, z_i)]^{2+m-n-\epsilon}. \tag{4}$$

Here $f(q_i, z_i)$ is a function of the external momenta and Feynman parameters, and $\epsilon = (d-4)/2$. Note that (4) contains an overall factor of ϵ. Hence, the expression will vanish in the limit $\epsilon \to 0$ unless one of the other factors generates a pole in ϵ.

[a] Such terms are generated because we do not expand the propagators in powers of μ^2. Hence, the numerators are effectively four-dimensional, while the denominators are d-dimensional. Accordingly, propagators are "cancelled" using the relation $L^2/\mathcal{L}^2 = 1 + \mu^2/\mathcal{L}^2$.

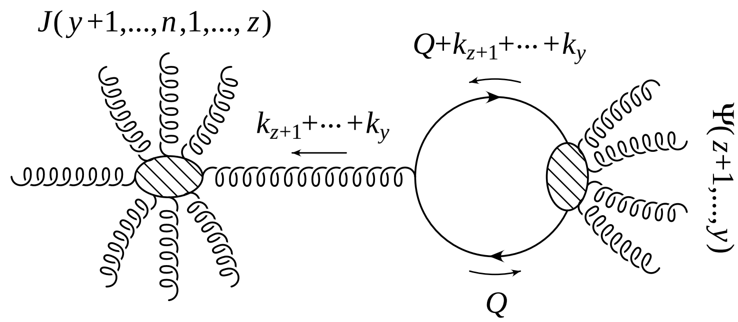

Fig. 1. Contributions to the n-gluon scattering amplitude involving a quark loop.

The first source of such a pole is the factor $\Gamma(m-\epsilon)$, which is singular if $m = 0$. This term, which has no powers of μ^2, is in some sense the "leading" term: it corresponds to the four-dimensional part of the numerator.

If $m \geq n-2$, then the factor $\Gamma(n-m-2+\epsilon)$ produces a pole. Power counting applied to such integrals reveals that they are ultraviolet divergent in four dimensions (μ^2 counts as two powers of loop momentum, just like L^2). Such terms occur only in the last stages of solution of the recursion relation, or in certain graphs with only a few legs attached to the loop. Furthermore, if $m = n-2$, the parameter integral is trivial to compute. This case comes up quite often, and the result is simple.[5]

The final potential source of poles in ϵ is the parameter integral. These are only infrared in nature. The exponent of $-f(q_i, z_i)$ is the one corresponding to a theory in $4+2m$ dimensions, where logarithmic infrared divergences disappear ($m \geq 1$). Thus, in the vast majority of cases, the parameter integral is finite.

We may summarize the above by the following two statements. First, integrals containing no numerator powers of μ^2 must always be computed. Second, integrals containing one or more powers of μ^2 contribute only if they would be ultraviolet divergent in four dimensions.

Armed with this knowledge, we are ready to proceed to a description of amplitude building based upon the recursion relation solutions. The first step is to "glue" one or more currents together with one or more of the vertices of the theory. For example, all of the contributions to the n-gluon amplitude that proceed through a quark loop are represented by Fig. 1, which consists of a double-off-shell quark current and a gluon current tied together at a single $q\bar{q}g$ vertex. Notice that in order to obtain all of the diagrams of this type, we must perform a sum over all of the ways to divide the n gluons between the two currents. Thus, the next step in the procedure is to perform as much of this algebra as possible. It is beneficial to watch for and take advantage of opportunities to reduce the number of propagators in each of the terms of the expression.

Once the contributions to the integrand are as simple as possible, we perform the momentum integration. The shift in loop momentum required for each term is determined and applied to the numerators. The parameter integrals are then performed using the differentiation method of Bern, Dixon and Kosower,[6] which allows integrals containing extra numerator factors of the Feynman parameters to be written as

derivatives of the corresponding scalar integral. The differentiation process is simple enough to let us prepare a table of such integrals. Given this table, it is straightforward to cancel any spurious divergences introduced in the reduction process. At this stage, all that remains is to "clean up" the result.

We now present two examples of results obtained by the above procedure. The first is for the process $\gamma\gamma \to \gamma \cdots \gamma$, for the helicity configuration $(--++\cdots+)$.[b] This calculation involves at worst box integrals—no higher point functions appear, even for an arbitrary number of photons. The result for even $n \geq 6$ reads

$$\mathcal{A}(1^-, 2^-, 3^+, \ldots, n^+) = -\frac{i(-e\sqrt{2})^n}{8\pi^2} \sum_{\mathcal{P}(3\ldots n)} \sum_{j=4}^{n} \frac{[\langle 1\ 3\rangle \langle j\ 3\rangle^* \langle j\ 2\rangle]^2}{\langle 3\ 4\rangle\langle 4\ 5\rangle \cdots \langle n\ 3\rangle} \frac{\Lambda(3,\ldots,j)}{(2k_3 \cdot k_j)^2}, \tag{5}$$

where the spinor inner products $\langle i\ j\rangle$ satisfy $\langle i\ j\rangle\langle i\ j\rangle^* = 2k_i \cdot k_j$. The function $\Lambda(3,\ldots,j)$ is a particular combination of dilogarithms given by

$$\begin{aligned}\Lambda(3,\ldots,j) \;=\; & \mathrm{Li}_2\left[1 - \frac{q^2(3,j)q^2(4,j-1)}{q^2(4,j)q^2(3,j-1)}\right] - \mathrm{Li}_2\left[1 - \frac{q^2(3,j)}{q^2(4,j)}\right] - \mathrm{Li}_2\left[1 - \frac{q^2(4,j-1)}{q^2(3,j-1)}\right] \\ & - \mathrm{Li}_2\left[1 - \frac{q^2(3,j)}{q^2(3,j-1)}\right] - \mathrm{Li}_2\left[1 - \frac{q^2(4,j-1)}{q^2(4,j)}\right] - \frac{1}{2}\ln^2\left[\frac{q^2(3,j-1)}{q^2(4,j)}\right],\end{aligned} \tag{6}$$

with $q(i,j) \equiv k_2 + k_i + \cdots + k_j$. Although it is not immediately obvious, Eq. 5 is indeed symmetric under the interchange $k_1 \leftrightarrow k_2$, as dictated by Bose statistics.

The second example is the expression for the quark-loop contributions to the scattering of n like-helicity gluons.[3] We find

$$\mathcal{A}(1^+, \ldots, n^+) = -\frac{i(-g\sqrt{2})^n}{48\pi^2} \sum_{\mathcal{P}(1\ldots n-1)} tr(T^{a_1} \cdots T^{a_n}) \sum_{j=2}^{n-2} \sum_{\ell=j+1}^{n-1} \frac{\mathrm{Tr}\{\bar{k}_j \kappa(1,j) \bar{\kappa}(1,\ell) k_\ell\}}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n\ 1\rangle}, \tag{7}$$

where $\kappa(1,j) \equiv k_1 + \cdots + k_j$.

In this talk, we have illustrated the use of recursion relations to simplify the computation of certain loop amplitudes containing an arbitrary number of external gauge bosons. Although there is still much work to be done before next-to-leading order cross sections may be extracted from these expressions, a great deal of progress has been made in this area, and real predictions are in sight on the horizon.

References

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[b] The helicity labels are always those for inward-directed momenta.

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